# Nonlinear effects in tunnelling escape in $N$-body quantum systems

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## Abstract

We consider the problem of tunneling escape of particles from a multiparticle system confined within a potential trap. The process is nonlinear due to the interparticle interaction. Using the hydrodynamic representation for the quantum equations of the multiparticle system we find the tunneling rate and time evolutions of the number of trapped particles for different nonlinearity values.

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The recent advances in the experiments on real Bose – Einstein condensates (BEC)[1,2,3] and nonlinear optical waves[4] have generated a huge body of works on the theoretical side. The standard approach on the theory side was the use of the nonlinear Schrödinger equation, with a potential term depending on the particle density (also known as the time dependent Gross-Pitaevskii equation[5,6,7,8,9]). However, as it became clear early in the investigation, the time behavior of such equations was very complex and rich. The phenomena of coherence[10], macroscopic tunneling[11], vortex formation,[12,13,14,15] instabilities, focusing and blowup are all new concepts which are related to the *nonlinear* nature of the systems. Most of the analysis of these hard and fundamental analytic problems are so far being dealt with by a combination of numerical schemes (e.g. Ref. 11) and finite dimensional phenomenological models. The quest for a theory that can adequately give the relevant nonlinear effects in the time dependent regime is therefore of major current interest.

In this letter we analyze the problem of resonance and tunneling of N-body Quantum systems by solving the corresponding nonlinear problem in the leading relevant approximation. Our approach combines ideas from Many body theory, nonlinear partial differential equations and resonance theory in Quantum Mechanics to offer a new unified approach to finding tunneling times for both linear and nonlinear systems. We focus on the problem of a BEC droplet in a potential well, that can tunnel through a finite barrier from the trap. We find the leading nonlinear corrections to the tunneling rate.

A droplet of a large number of atoms, $N$, with boson statistics is confined by an external potential $V_{ext}(\mathbf{R})$. The multiparticle wave function describing the quantum state of such a system satisfies the Schrödinger equation

$$i\hbar\frac{\partial\Psi(\{\mathbf{R}_i\})}{\partial t} = \left(-\frac{\hbar^2}{2m}\sum_i\frac{\partial^2}{\partial\mathbf{R}_i^2} + \sum_i V_{ext}(\mathbf{R}_i) + \frac{1}{2}\sum_{i\neq j}V_{int}(\mathbf{R}_i - \mathbf{R}_j)\right)\Psi(\{\mathbf{R}_i\}) \quad (1)$$

Starting from this equation we may derive equations of motion for moments of the single particle Wigner function $\rho_W(\mathbf{p}, \varepsilon, \mathbf{R}, t)$.[18] We concentrate here on the condensate behavior, which is supposed to be separated from the dissipative excitations (a detailed discussion of this separation can be found in Ref. 17). The derivation can be made gauge invariant and its details are presented in the papers[16] where many relevant references can be also found. The most important of the moments are the density distribution of the particles

$$\rho(\mathbf{R}, t) = \frac{1}{(2\pi\hbar)^4}\int d^3p\,d\varepsilon\,\rho_W(\mathbf{p}, \varepsilon, \mathbf{R}, t)$$

and the velocity field

$$\mathbf{v}(\mathbf{R}, t) = \frac{1}{(2\pi\hbar)^4} \frac{1}{\rho(\mathbf{R}, t)} \int d^3 p \, d\varepsilon \, \mathbf{p} \rho_W(\mathbf{p}, \varepsilon, \mathbf{R}, t)$$

for which the continuity equation

$$\frac{\partial \rho(\mathbf{R}, t)}{\partial t} + \nabla(\mathbf{v}(\mathbf{R}, \mathbf{t})\rho(\mathbf{R}, t)) = 0 \tag{2}$$

and the Euler-type equation

$$\frac{\partial \mathbf{v}(\mathbf{R}, \mathbf{t})}{\partial t} + (\mathbf{v}(\mathbf{R}, \mathbf{t}) \cdot \nabla)\mathbf{v}(\mathbf{R}, \mathbf{t}) = -\frac{1}{m} \nabla(V_{ext}(\mathbf{R}) + V_{qu}(\mathbf{R}) + V_{eff}(\mathbf{R})). \tag{3}$$

are obtained. The velocity field is not necessarily potential and three dimensional cases allows for vortices. Here the "quantum potential" $V_{qu}(\mathbf{R}) = -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho(\mathbf{R}, t)}}{\sqrt{\rho(\mathbf{R}, t)}}$ accounts for the quantum character of the liquid in the droplet, whereas $V_{eff}(\mathbf{R})$ is the contribution due to the inter-particle interaction. In many cases, the interaction potential $V_{eff}(\mathbf{R})$ can be represented as a functional of the particle density $\rho(\mathbf{R}, t)$. For example, if we assume that the inter-particle interaction is point-like, $V_{int}(\mathbf{R}_i - \mathbf{R}_j) = \frac{1}{2}\lambda\delta(\mathbf{R}_i - \mathbf{R}_j)$ then the direct calculations, whose many aspects are parallel to those presented in Ref.[17], result in the lowest order in $\lambda$ in $V_{eff}(\mathbf{R}) = \lambda\rho(\mathbf{R}, t)$. Then equations (2) and (3) become equivalent to the time dependent Gross-Pitaevskii equation. Higher order term in $\lambda$ (to be discussed separately) will produce a correction to the above effective potential $V_{eff}(\mathbf{R})$ as well as a dissipative contribution due to the interaction with the excitations above the condensate.

In this presentation we will limit ourselves to discussion of the problem of tunneling evaporation of the droplet kept within a one-dimensional potential depicted in figure 1. It means that we will have to consider the one dimensional versions of equations (2) and (3), which read now

$$\rho_t(x, t) + \frac{\partial}{\partial x}[\rho(x, t)v(x, t)] = 0. \tag{4}$$

and

$$v_t(x, t) + v(x, t)v_x(x, t) = \frac{1}{m} \frac{\partial}{\partial x} \left[ V_{ext}(x) - \frac{1}{\sqrt{\rho(x, t)}} \frac{\hbar^2}{2m} \sqrt{\rho(x, t)} + U_{eff}(\rho(x, t)) \right]. \tag{5}$$

In order to solve the problem we first introduce an auxiliary confining potential $V_0(x)$, which coincides with the real potential $V_{ext}(x)$ for small $x$ but differs from it (dashed line

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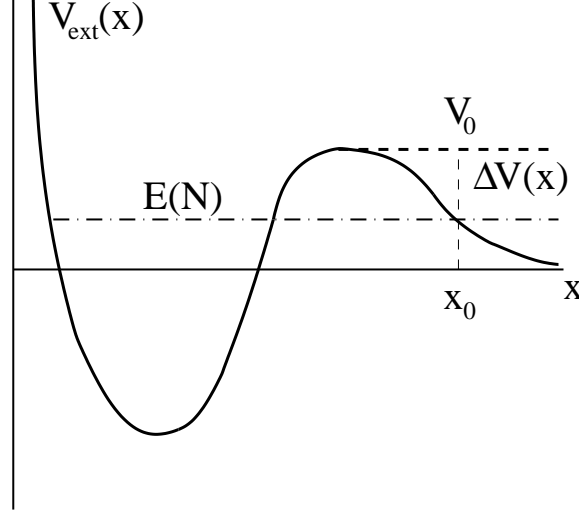
FIG. 1: Potential well keeping the droplet. The dashed line shows the auxiliary potential $V_0(x)$ at larger $x$. The dash dotted line corresponds to the energy $E(N)$ for a given number of particles $N$.

in fig. 1) at large $x$. Stationary states are possible in such potential with the density distribution following from equation (5) at $v(x,t) = 0$ and with $V_0(x)$ substituted for $V_{ext}(x)$.

$$V_0(x)\sqrt{\rho(x,t)} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\sqrt{\rho(x,t)} + U_{eff}(\rho(x,t))\sqrt{\rho(x,t)} = E(N)\sqrt{\rho(x,t)} \qquad (6)$$

$E(N)$ is the single particle eigen-energy in a droplet with $N$ particles. One readily recognizes a stationary Schrödinger-type equation for the single particle wave function $\psi(x,t) = \sqrt{\rho(x,t)}$ and with the effective potential $U_{eff}(\rho(x,t))$ accounting for the many body effects. In case of a contact interaction between the particles equation (6) becomes celebrated stationary Gross-Pitaevskii equation.

The solution of equation (6) at a given initial number of particles $N_0$ is considered as an initial state of a droplet in the real external potential $V_{ext}(x)$. In this case the density distribution $\rho(x,t)$ becomes time dependent and its dynamics is governed by equation (5). Here we will follow the time evolution of the number of particles $N(t)$ within the droplet, defined as $N(t) = \int_0^{x_0} dx\rho(x,t)$ and with $x_0$ chosen as the exit point of the potential $V_{ext}(x)$ at a given energy $E(N)$ (see fig. 1). $N(t)$ at $t = 0$ nearly coincides with the initial number of particle $N_0$ since only exponentially small part of the particle density remains outside the potential well at $x > x_0$.

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Equation (5) can be solved if we assume that the velocity field and the density distribution change adiabatically slow with time. The applicability of this approximation will be discussed below. Then we may neglect the time derivative $v_t(x,t) = 0$ in the rhs of (5). Using the fact that initially the density distribution $\rho(x,t)$ satisfies equation (6) we may find the velocity field $v(x,t) = \sqrt{\frac{2}{m}\Delta V(x)}$ where $\Delta V(x) = V_0(x) - V_{ext}(x)$

Now we find the number of particles within the well $V_{ext}(x)$ by integrating the continuity equation (4) from 0 to the exit point $x_0$. According to our approximation both the density and the velocity fields vary slowly with time only due to the time variation of the number of particles $N(t)$. The exit point $x_0$ is connected with eigen-energy $E(N)$ of the tunneling particles, so that the velocity becomes $v(x_0,t) = \sqrt{\frac{2}{m}[V_0 - E(N(t))]}$ and we get equation

$$t = -\int_{N_0}^{N} d\tilde{N} \frac{1}{\rho(x_0)\sqrt{\frac{2}{m}[V_0 - E(\tilde{N})]}}. \tag{7}$$

which determines implicitly the time dependence of the number of particle $N(t)$.

In order to estimate the particle density $\rho(x_0)$ at the exit point the particle density at the eigen energy $E(N)$ is written as $\rho(x;N) = N\varrho(x;N)$, with $\varrho(x;N)$ normalized to one. $\rho(x,N)$ satisfies Schrödinger equation (6) but we need to know its value only at the exit point where it is exponentially small and we may neglect the inter-particle interaction. As a result we get $\sqrt{\varrho(x_0;N(t))} \propto e^{-\alpha x_0}$ with $\alpha = \frac{1}{\hbar}\sqrt{2m(V_0 - E(N))}$.

If the dependence of the energy $E(N)$ on the number of particles in the droplet is known, equation (7) can be solved and the time dependence $N(t)$ of the number of particles can be found. As an example we carry out the calculation for the contact inter-particle interaction when equations (4) and (5) correspond to the Gross-Pitaevskii equation. At not too large $N$ we may approximately assume a linear dependence of the energy on the number of particles $E(N) = E(0) - \tilde{\lambda}N$[19] and represent the integral (7) in the form

$$\frac{1}{2}\Gamma_0 t = -\int_{y_0}^{y} dy \frac{e^{\frac{1}{2}\alpha_0 x_0 y^2}}{y\sqrt{1+y^2}} \tag{8}$$

where $y^2 = 2\frac{N}{N_{GP}}$, $N_{GP} = \frac{\hbar^2 \alpha_0^2}{\tilde{\lambda}m}$, $\alpha_0 = \frac{1}{\hbar}\sqrt{2m(V_0 - E(0))}$ and $\Gamma_0 = \frac{\alpha_0 \hbar}{m}\varrho(x_0;0)$.

Using the property

$$\frac{1}{y\sqrt{1+y^2}} = \frac{d}{dy}\ln\frac{y+\sqrt{y^2+1}-1}{y+\sqrt{y^2+1}+1}$$

the integral (8) is taken by parts and after that the slow logarithmic function is taken out of integral. As a result we get

$$-\frac{1}{2}\Gamma_0 t = \left[\ln\frac{y+\sqrt{y^2+1}-1}{y+\sqrt{y^2+1}+1} - \ln\frac{y_0+\sqrt{y_0^2+1}-1}{y_0+\sqrt{y_0^2+1}+1}\right]e^{\frac{1}{2}\alpha_0 x_0 y^2}.\tag{9}$$

This equation can be solved iteratively. First we assume that $y = y_0$ in the exponential function equation (9) and get

$$y_1(t) = \frac{2f(t;y_0)}{1-f^2(t;y_0)}\tag{10}$$

where $f(t;y) = \nu(y)e^{-\frac{1}{2}\Gamma_0 e^{-\alpha_0 x_0 y^2}t}$ and $\nu(y) = \frac{y+\sqrt{1+y^2}-1}{y+\sqrt{1+y^2}+1}$, or $y = \frac{2\nu(y)}{1-\nu(y)^2}$. At the next iteration we substitute solution (10) into the function $f(t;y)$ and repeat the procedure,

$$N(t) = \frac{\hbar^2\alpha_0^2}{2m\widetilde{\lambda}}\left(\frac{2f(t;y_1(t))}{1-f^2(t;y_1(t))}\right)^2\tag{11}$$

The function $\nu$ in the above equations is always less than one. $\nu$ tends to zero in the linear limit, when $\lambda \to 0$, and one gets the exponential decay $N(t) = N_0 e^{-\Gamma_0 t}$. At a large nonlinearity parameter $\lambda$, the function $\nu$ may be close to one and the initial decay strongly deviates from the exponential behavior. However, $\nu$ diminishes in the time course and at large time the asymptotical behavior of the number of particle $N(t)$ becomes exponential.

Before discussing the results it is worthwhile to estimate the validity of the adiabatic approximation made while solving equation (5). For this we need to compare the time derivative of the velocity $v_t$ near the exit point $x_0$, neglected in the above calculations, with the right hand side of equation (5). The time dependence of the velocity results mainly from the time dependence of the exit point, so that

$$v_t = \frac{2\dot{x}_0}{mv(x_0)}V'_{ext}(x)|_{x=x_0} \approx \frac{1}{mv(x_0)}\frac{V_0}{l}\dot{x}_0\tag{12}$$

with $l$ being the typical length characterizing the potential $V_{ext}(x)$. Since $x_0$ is the solution of equation $V_{ext}(x) = E(0) - \widetilde{\lambda}N(t)$ we get $\dot{x} \approx \frac{\dot{N}}{N}\frac{\widetilde{\lambda}N}{V_0}l$. The right hand side of equation (5) can be estimated as $\approx V_0/ml$ so that comparing it with the time derivative of the velocity field (12) we need to check the inequality

$$\frac{v_t ml}{V_0} \approx \frac{\dot{x}_0}{v(x_0)} \approx \tau_{tr}\frac{\dot{N}}{N}\frac{\widetilde{\lambda}N}{V_0} \ll 1.\tag{13}$$

Here $\tau_{tr} = l/v$ is so called traversal time, which roughly corresponds to the time needed for a tunneling particle to traverse the classically forbidden underbarrier region. It is quite
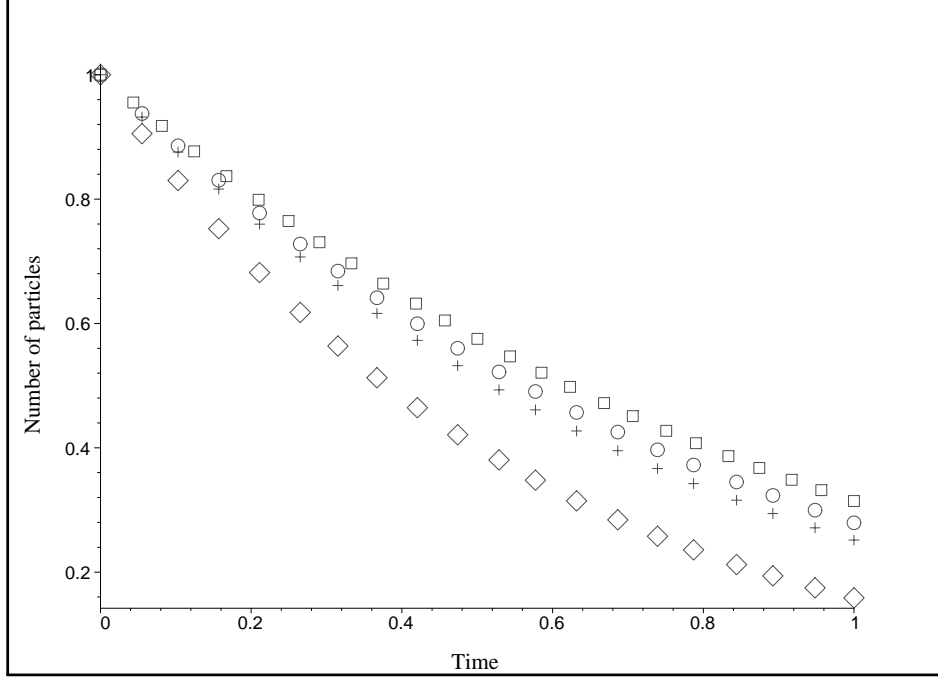
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FIG. 2: Relative number of particles $(N(t)/N_0)$ as a function of time (in units $1/\Gamma$) for different values of the initial number of particles $N_0/N_{GP}$ and $\alpha_0 x_0 = 3$: diamonds — $N_0/N_{GP} = 0$; crosses — $N_0/N_{GP} = 0.004$; circles — $N_0/N_{GP} = 0.04$; boxes — $N_0/N_{GP} = 0.4$

clear that at any reasonably high potential barrier the tunneling rate is small enough so that $\tau_{tr}\dot{N}/N \ll 1$. The ratio $\widetilde{\lambda}N/V_0$ cannot be large and in many realistic cases it is even small at any reasonable number of particles in the droplet. We may conclude that the inequality (13) is robust and holds at all reasonable parameters of the system, which justifies the adiabatic approximation applied in this paper.

Figure 2 shows the time dependence of the number of particle $N(t)$ in a droplet for different initial numbers $N_0$. These numbers are scaled with the quantity $N_{GP}$ defined in (8), which can be thought of as a number of particles, at which the nonlinear contribution to the single particle energy $E(N)$ becomes comparable with the linear part $E(0)$. We see that the time decay of the droplet is purely exponential, if the initial number $N_0$ is small. It deviates from this exponential behavior and becomes essentially slower at large initial numbers of particles in the droplet. This deviation is stronger the larger $N_0$ is, but at large time when the total number of particles $N(t)$ becomes smaller the behavior of the decay curve tends to the exponential one.

We may also find (directly from (8)) the initial slope of the decay curve as a function of
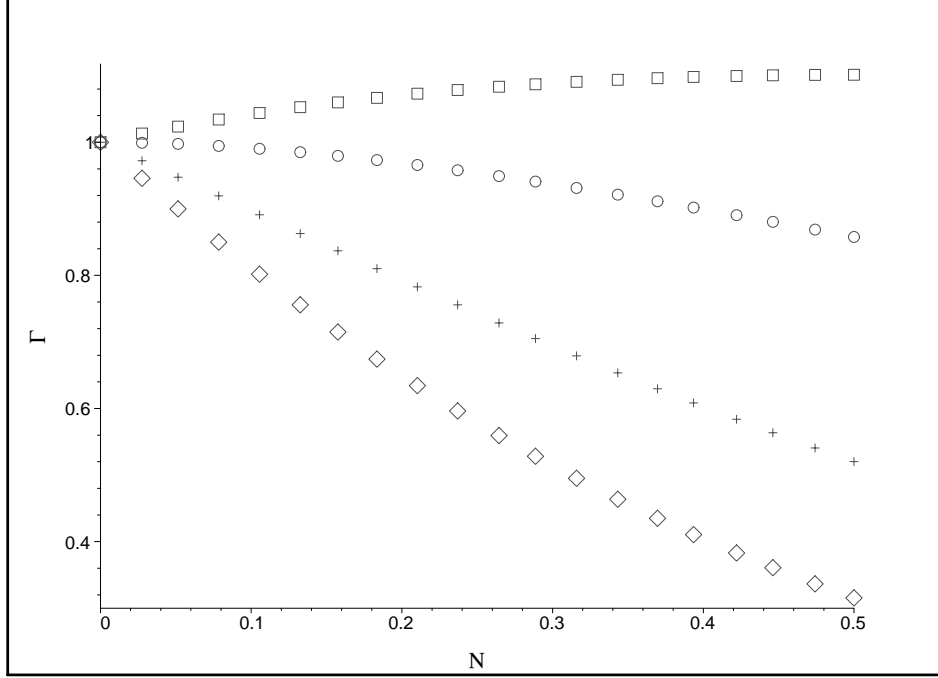
FIG. 3: Initial rate of evaporation $\Gamma$, in units of $\Gamma_0$ as a function of the initial number of particles $N_0/N_{GP}$ at for several choices of $\alpha_0 x_0$: diamonds — $\alpha_0 x_0 = 3$ ; crosses — $\alpha_0 x_0 = 2$ ; circles — $\alpha_0 x_0 = 1$ ; boxes — $\alpha_0 x_0 = 0.5$

the initial number of particles $N_0$,

$$-\frac{1}{N\Gamma_0}\frac{dN}{dt}\bigg|_{t=0} = \sqrt{1+2\frac{N}{N_{GP}}}\ e^{-\alpha_0 x_0 \frac{N}{N_{GP}}}. \tag{14}$$

It is interesting to note that the nonlinear effects usually cause a decrease of the escape rate with the increasing initial number of particles $N_0$ in the droplet as, e.g. is demonstrated in fig. 2. On one hand, this trend is determined by the exponential function in (14), corresponding to the density of particles at the exit point $x_0$, and the decrease is stronger for higher and wider barriers, large $\alpha x_0$. On the other hand the square root factor, i.e. the particle velocity at the exit point, may result in an increase of the escape rate. The latter may happen for rather low barriers when $\alpha_0 x_0 < 1$ (see fig. 3).

We have demonstrated here how the hydrodynamic approach to the description of a multiparticle quantum system lead to a solution of the nonlinear problem of particle tunneling escape from a trap. Using the adiabatic approximation (neglect of $v_t$), justified for nonlinearities, which are not necessarily small, we succeeded in obtaining an analytical solution in a one-dimensional case for the time dependence of the number of trapped particles and for

the tunneling rates at different nonlinearity values. The technique, we use, goes beyond the WKB approximation and, in particular, divergencies at the turning point of the trap potential do not appear. That is why applying the same technique directly to three dimensional systems seems to be straightforward. Contrary to the standard WKB approach velocity is not necessarily a potential field (see, e.g. discussion in Ref. 16) and in three dimensional cases vortices can be considered.

[1] M.H.J. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, Science **269**, 198 (1995).

[2] K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 39693973 (1995).

[3] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. 75 , 1687 (1995).

[4] J.W. Fleischer, M. Segev, N.K. Efremidis, D.B. Christodoulides, Nature **422**,6928 (2003).

[5] S. Inouye, M. R. Andrews, J. Stenger, H. J. Miesner, D. M. Stamper-Kurn, and W. Ketterle, Nature **392**, 151 (1998).

[6] E. H. Lieb and J. Yngvason, Phys. Rev. Lett. 80 , 2504 (1998); E. H. Lieb, R. Seiringer and J. Yngvason, Phys. Rev. A 61 , 043602 (2000).

[7] L.P. Pitaevskii, Sov. Phys. JETP **13**, 451 (1961).

[8] E.P. Gross, J.Math.Phys. **4** , 195 (1963).

[9] C. Sulem and P. L. Sulem, The Nonlinear Schrödinger Equation (Springer-Verlag, New York, 1999).

[10] S.L. Rolston and W.D. Phillips, Nature **416**, 219 (2002)

[11] Y. Shin, M. Saba, T.A. Pasquini, W. Ketterle, D. E. Pritchard, and A.E. Leanhardt, Phys. Rev. Lett. **92**, 050405 (2004).

[12] J.E. Williams and M.J. Holand, Nature **568**, 401 (1999).

[13] M.R. Matthews, B.P. Anderson, P.C. Haljan, D.S. Hall, C.E. Wieman, and E.A. Cornell, Phys.Rev.Lett. **83**, 2498 (1999).

[14] K.W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys.Rev.Lett. **84**, 806 (1999)

[15] S. Inouye, S. Gupta, T. Rosenband, A.P. Chikkatur, A. Gorlitz, T.L. Gustavson, A.E. Leanhardt, D.E. Pritchard, and W. Ketterle, Phys.Rev.Lett. **87**, 080402 (2001).

[16] M. Levanda and V. Fleurov, Ann.Phys. NY **292**, 199 (2001); ibid, J.Phys.: Condens Matter **6**, 7889 (1994)

[17] M. Imamov-Tomasovic and A. Griffin, J.Low Temp.Phys. **122**, 617 (2001)

[18] Single particle Wigner function may be obtained by integrating the coordinates and momenta of $N-1$ particles of an $N$ particle Wigner functions. A more detailed description can be found in[16].

[19] A. Soffer and M.I. Weinstein, "Selection of the Ground state...", Preprint 2001, to appear